Solving 3 Simultaneous Equations using Matrices

We have the **3** simultaneous equations:

- $\boldsymbol{3x - 2y + z = 7}$
- $\boldsymbol{2x + y - 3z = -1}$
- $\boldsymbol{x + 4y + 2z = 12}$

We must solve them using matrices

Step 1

First, we put the equations into **matrices format**

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 12 \end{pmatrix}$$

Step 2

Then we need to work out the **transposition of the 3x3 matrix** as the first step towards finding the minor, cofactor and the inverse (important steps in solving our problem). This matrix is called the **adjugate matrix**. To make this matrix **we flip the positions around a diagonal mirror**, this means we essentially take the coordinate of each value in the matrix and flip it to become its opposite. i.e. (i, j) becomes (j, i). **We can also easily visualise this by taking our individual rows and turning them into columns instead.**

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & 4 \\ 1 & -3 & 2 \end{pmatrix}$$

Step 3

Our next step is to find our **matrix of minors**. This involves finding a “minor matrix” for every value in our **adjugate matrix**. To find a minor matrix we look at **everything which isn’t in the row/column** that the value we’re look at is in and use that as a 2x2 matrix.

So for example, in our matrix we have $\begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & 4 \\ 1 & -3 & 2 \end{pmatrix}$.

For our first value (highlighted in red) we rule out our first row and column $\begin{pmatrix} 3 & \text{[redacted]} & \text{[redacted]} \\ \text{[redacted]} & 1 & 4 \\ \text{[redacted]} & -3 & 2 \end{pmatrix}$.

What we are left with is a **2x2 matrix (our minor)** $\begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix}$ which we can then work out the determinant of using our 2x2 **determinant rule**: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (A)(D) - (B)(C)$.

So, for our example which we found above: $\begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} = (1)(2) - (4)(-3) = 14$

$\begin{pmatrix} 3 & \text{[redacted]} & \text{[redacted]} \\ \text{[redacted]} & 1 & 4 \\ \text{[redacted]} & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -3 & 2 \end{pmatrix} = 14$	$\begin{pmatrix} \text{[redacted]} & 2 & \text{[redacted]} \\ -2 & \text{[redacted]} & 4 \\ 1 & \text{[redacted]} & 2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & 2 \end{pmatrix} = -8$	$\begin{pmatrix} \text{[redacted]} & \text{[redacted]} & 1 \\ -2 & 1 & \text{[redacted]} \\ 1 & -3 & \text{[redacted]} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} = 5$
$\begin{pmatrix} \text{[redacted]} & 2 & 1 \\ -2 & \text{[redacted]} & \text{[redacted]} \\ \text{[redacted]} & -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} = 7$	$\begin{pmatrix} 3 & \text{[redacted]} & 1 \\ \text{[redacted]} & 1 & \text{[redacted]} \\ 1 & \text{[redacted]} & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 5$	$\begin{pmatrix} 3 & 2 & \text{[redacted]} \\ \text{[redacted]} & \text{[redacted]} & 4 \\ 1 & -3 & \text{[redacted]} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & -3 \end{pmatrix} = -11$
$\begin{pmatrix} \text{[redacted]} & 2 & 1 \\ \text{[redacted]} & 1 & 4 \\ 1 & \text{[redacted]} & \text{[redacted]} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = 7$	$\begin{pmatrix} 3 & \text{[redacted]} & 1 \\ -2 & \text{[redacted]} & 4 \\ \text{[redacted]} & -3 & \text{[redacted]} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} = 14$	$\begin{pmatrix} 3 & 2 & \text{[redacted]} \\ -2 & 1 & \text{[redacted]} \\ \text{[redacted]} & \text{[redacted]} & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} = 7$

This gives us the **minor matrix** of $\begin{pmatrix} 14 & -8 & 5 \\ 7 & 5 & -11 \\ 7 & 14 & 7 \end{pmatrix}$

Step 4

We can combine this new **minor matrix** with a pattern to finally make our **cofactor matrix**. Note a + means the sign stays the same and a – means the sign flips. This cofactor matrix is what we will utilise to solve the rest of the problem.

$$\begin{pmatrix} 14 & -8 & 5 \\ 7 & 5 & -11 \\ 7 & 14 & 7 \end{pmatrix} * \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} 14 & 8 & 5 \\ -7 & 5 & 11 \\ 7 & -14 & 7 \end{pmatrix}$$

Step 5

Next we need to **work out the determinant** as we have all other values to fill our final equation. Working out of the determinant just involves looking at both our **adjugate matrix** and our **cofactor matrix.**

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & 4 \\ 1 & -3 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 14 & 8 & 5 \\ -7 & 5 & 11 \\ 7 & -14 & 7 \end{pmatrix}$$

We can pick and row or column we choose and then we just multiply the adjugate value by its corresponding cofactor value and add them all up.

For example, if I pick Row 1:

$$B = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & 4 \\ 1 & -3 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 14 & 8 & 5 \\ -7 & 5 & 11 \\ 7 & -14 & 7 \end{pmatrix}$$

So $(3 * 14) + (2 * 8) + (1 * 5) = 63$, so our **determinant** is **63**

Note any row or column should give you the same answer

Step 6

We can now write out our final equation to solve.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{63} \begin{pmatrix} 14 & 8 & 5 \\ -7 & 5 & 11 \\ 7 & -14 & 7 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 12 \end{pmatrix}$$

Step 7

Our first step in solving is to multiply out the right-hand side of the equation **(multiply the two matrices).** As it is a 3x3 by a 3x1 we will end up with a 3x1 matrix. To achieve this, we multiply each **column** in the 3x3 by each **row** in the 3x1.

$$\begin{pmatrix} 14 & 8 & 5 \\ -7 & 5 & 11 \\ 7 & -14 & 7 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 14 * 7 + 8 * -1 + 5 * 12 \\ -7 * 7 + 5 * -1 + 11 * 12 \\ 7 * 7 + -14 * -1 + 7 * 12 \end{pmatrix} = \begin{pmatrix} 150 \\ 78 \\ 147 \end{pmatrix}$$

Step 8

So now we have an equation which we can **simplify** to get our answers.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{63} \begin{pmatrix} 150 \\ 78 \\ 147 \end{pmatrix} = \begin{pmatrix} 50/21 \\ 26/21 \\ 7/3 \end{pmatrix} \approx \begin{pmatrix} 2.380952381 \\ 1.238095238 \\ 2.333333333 \end{pmatrix}$$

So: $\mathbf{x \approx 2.381 \quad y \approx 1.238 \quad z \approx 2.333}$